Jump

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考虑跳系统，带有 Markov 跳或者 Levy process 驱动的系统。跳跃项可以通过 Poisson random measure 表示，于是有 Poisson integral。

Poisson integral 这个概念我一直不太理解，所以遇到了一些问题。

1. Levy process 如何用 random measure 表示？

2. 用 Poisson integral 表示的跳跃部分，如何计算跳跃部分的二次变差？我尝试用指数鞅不等式（Doob 的变形）+ Broel-Cantelli 引理来估计跳跃部分的一个上界。

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1 Levy process 随机测度表示

We consider now a special case of Levy processes: 'a Brownian motion with drift and Poisson jumps'. More precisely, let

$$H_t = mt + \sigma W_t + \sum_{k=1}^{N_t} \xi_k,$$

where $W = (W_t)_{t\geq 0}$ is a Wiener process (Brownian motion), $\xi_1, \xi_2, \cdots$ are independent identically distributed random variables with distribution function

$F(x) = \mathbb{P}(\xi_1 \leq x)$, and $N = (N_t)_{t\geq 0}$ is the standard Poisson process with parameter $\lambda > 0 (\mathbb{E}N_t = \lambda t)$.

We assume that $W, N, (\xi_1, \xi_2, \cdots)$ are jointly independent.

The following chain of relation brings us easily to the canonical representation, giving us the triplet of predictable characteristics:

$$
\begin{aligned}
H_t =& mt + \sigma W_t + \sum_{k=1}^{N_t} \xi_k & (1)\\
=& mt + \sigma W_t + \int_0^t \int x d\mu & (2)\\
=& (mt + \int_0^t \int g(x) d\nu) + (\sigma W_t + \int_0^t \int g(x) d(d\mu - d\nu)) & (3)\\
&+ \int_0^t \int (x - g(x)) d\mu & (4)\\
=& t(m + \int_0^t g(x) F(dx)) + (\sigma W_t + \int_0^t \int g(x) d(\mu - \nu)) & (5)\\
&+ \int_0^t \int (x - g(x)) d\mu. & (6)
\end{aligned}
$$

Hence

$$
\begin{aligned}
B(g)t &= t(m + \int_0^t g(x) F(dx)), & (7)\\
C_t &= \sigma^2 t, & (8)\\
d\nu &= \lambda dt F(dx) & (9)
\end{aligned}
$$

Ref to Page 671, Shiriaev, A. N. Essentials of Stochastic Finance: Facts, Models, Theory. Advanced Series on Statistical Science & Applied Probability, v. 3. Singapore; River Edge, N.J: World Scientific, 1999.

(1) 是常见的 Levy process，

(2) 用 random measure 表示，

(3) 使用截短函数保证 $\int_0^t \int g(x) d\nu$ 良定。但是，(4)，即 (7) 是怎么推导的？

2 指数鞅不等式

Theorem Let $g = (g_1, \cdots, g_m) \in \mathcal{L}^2 (R_+; R^{1\times m})$, and let T, α, β be any positive numbers. Then

$$
P\left\{ \sup_{0\le t\le T} \left[\int_0^t g(s) dB(s) - \frac{\alpha}{2} \int_0^t |g(s)|^2 ds \right] > \beta \right\} \le e^{-\alpha\beta}
$$

Proof. For every integer $n \geq 1$, define the stopping time

$$\tau_n = \inf\left\{t \geq 0 : \left|\int_0^t g(s)dB(s)\right| + \int_0^t |g(s)|^2 ds \geq n\right\}$$

and the Itô process

$$x_n(t) = \alpha \int_0^t g(s)I_{[[0,\tau_n]]}(s)dB(s) - \frac{\alpha^2}{2}\int_0^t |g(s)|^2 I_{[[0,\tau_n]]}(s)ds.$$

Clearly, $x_n(t)$ is bounded and $\tau_n \uparrow \infty$ a.s. Applying the Itô's formula to $\exp[x_n(t)]$ we obtain that

$$\begin{aligned}\exp[x_n(t)] &= 1 + \int_0^t \exp[x_n(s)]dx_n(s) + \frac{\alpha^2}{2}\int_0^t \exp[x_n(s)]|g(s)|^2 I_{[[0,\tau_n]]}(s)ds \\ &= 1 + \alpha\int_0^t \exp[x_n(s)]g(s)I_{[[0,\tau_n]]}(s)dB(s)\end{aligned} \tag{10}$$

In view of Theorem 5.21, one sees that $\exp[x_n(t)]$ is a nonnegative martingale on $t \geq 0$ with $E(\exp[x_n(t)]) = 1$.

Hence, by Theorem 3.8, we get that

$$P\left\{\sup_{0\leq t\leq T} \exp[x_n(t)] \geq e^{\alpha\beta}\right\} \leq e^{-\alpha\beta}E(\exp[x_n(T)]) = e^{-\alpha\beta}$$

That is,

$$P\{\sup_{0\leq t\leq T}[\int_0^t g(s)I_{[[0,\tau_n]]}(s)dB(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2 I_{[[0,\tau_n]]}(s)ds] > \beta\} \leq e^{-\alpha\beta}$$

Now the required follows by letting $n \to \infty$ and the proof is therefore complete.

考虑如下跳系统，$(X(\cdot), r(\cdot))$ 是 Markov process。$N(t,z)$ is a $\mathcal{F}_t$ -adapted Poisson random measure on $[0,+\infty) \times \mathbb{R}$ with a σ -finite intensity measure $\pi(du)$, and then the compensator martingale measure $\widetilde{N}(t,u)$ satisfies $\widetilde{N}(dt,dz) = N(dt,dz) - \pi(du)dt$.

$$\begin{aligned}\mathrm{d}X(t) =& f(X(t), X(t-\tau(t)), t, r(t))\mathrm{d}t + g(X(t), X(t-\tau(t)), t, r(t))\mathrm{d}B(t)\\ &+ \int_U h(X(t-), X(t-\tau(t)), t, r(t-), u)N(\mathrm{d}t, \mathrm{d}u)\end{aligned} \tag{11}$$

$$P\{r(t+\Delta) = j \mid r(t) = i, X(t) = x\} = \begin{cases} q_{ij}(x)\Delta + o(\Delta), & \text{if } j \neq i;\\ 1 + q_{ii}(x)\Delta + o(\Delta), & \text{if } j = i, \end{cases} \tag{12}$$

定义如下算子

$$\begin{aligned}\mathcal{L}V(x,y,t,i) =& \frac{\partial V(x,t,i)}{\partial t} + \sum_{k=1}^{n} f^k(x,y,t,i)\frac{\partial}{\partial x_k}V(x,t,i)\\ &+ \frac{1}{2}\sum_{k,l=1}^{n}\sum_{r=1}^{d} g^{kr}(x,y,,t,i)g^{lr}(x,y,t,i)\frac{\partial^2}{\partial x_k \partial x_l}V(x,y,t,i)\\ &+ \int_U [V(x+h(x,y,t,i,u),t,i) - V(x,t,i)]\pi(du)\\ &+ \sum_{j\in\mathbb{M}} q_{ij}(x)V(x,t,j),\end{aligned} \tag{13}$$

$$\mathcal{Q}_{\mathcal{B}}V(x,y,t,i) = \sum_{k,l=1}^{n}\sum_{r=1}^{d} g^{kr}(x,y,,t,i)g^{lr}(x,y,t,i)\frac{\partial}{\partial x_k}V(x,y,t,i)\frac{\partial}{\partial x_l}V(x,y,t,i),$$

$$\tag{14}$$

$$\mathcal{Q}_{\mathcal{M}_\in}V(x,y,t,i) = \sum_{j\neq i, j\in\mathbb{M}} q_{ij}(x)|V(x,t,j) - V(x,t,i)|^2, \tag{15}$$

$$\mathcal{Q}_{\mathcal{M}_\ni}V(x,y,t,i) = \int_U \lambda|V(x,t,j) - V(x,t,i)|^2\pi(du), \tag{16}$$

Ito 公式如下

$$V(X(t),t,\alpha(t)) = V(x_0,0,i_0) + \int_0^t \mathcal{L}V(X(s),X(s-\delta(s)),s,\alpha(s))ds \tag{17}$$
$$+ M_1(t) + M_2(t) + M_3(t) \tag{18}$$

其中

$$M_1(t) = \int_0^t \sum_{i=1}^{r}\sum_{k=1}^{d} g^{ik}(X(s),X(s-\delta(s)),s,\alpha(s))\frac{\partial}{\partial x_i}V(X(s),s,\alpha(s))dW_s^k \tag{19}$$

$$M_2(t) = \int_0^t \int_{\mathbb{R}} \left[V\left(X\left(s^-\right),s^-,\alpha\left(s^-\right)+h\left(X\left(s^-\right),\alpha\left(s^-\right),z\right)\right)\widetilde{N}_1(ds,dz)\right. \tag{20}$$

$$M_3(t) = \int_0^t \int_U [V(X(s^-)+h(X(s^-),X(s^- - \tau(s)),s^-,r(s),u))]\widetilde{N}(ds,du). \tag{21}$$

试图使用指数鞅不等式

$$P\left\{\omega : \sup_{0\le t\le w_1}\left[M_1(t) - \int_0^t \frac{u_1}{2}\mathcal{Q}_{\mathcal{B}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\right] > v_1\right\} \le e^{-u_1v_1}, \tag{22}$$

$$P\left\{\omega : \sup_{0\le l\le w_2}\left[M_2(t) - \int_0^t \frac{u_2}{2}\mathcal{Q}_{\mathcal{M}_\in}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\right] > v_2\right\} \le e^{-u_2v_2}, \tag{23}$$

$$P\left\{\omega : \sup_{0\le t\le w_3}\left[M_3(t) - \int_0^t \frac{u_3}{2}\mathcal{Q}_{\mathcal{M}_\ni}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\right] > v_3\right\} \le e^{-u_3v_3} \tag{24}$$

$$\tag{25}$$

其中，$w_i, u_i, v_i, i = 1,2,3$ 为正数.

在指数鞅不等式中，有：

$$[M_1, M_1](t) = \int_0^t \mathcal{Q}_{\mathcal{B}} V(X(s), X(s-\delta(s)), s, \alpha(s)) ds, \tag{26}$$

$$[M_2, M_2](t) = \int_0^t \mathcal{Q}_{\mathcal{M}_\in} V(X(s), X(s-\delta(s)), s, \alpha(s)) ds, \tag{27}$$

$$[M_3, M_3](t) = \int_0^t \mathcal{Q}_{\mathcal{M}_\ni} V(X(s), X(s-\delta(s)), s, \alpha(s)) ds. \tag{28}$$

1. 首先不太理解 $[M_2, M_2](t)$ 这一项的估计，不会计算 $\mathcal{Q}_{\mathcal{M}_\in} V(x, y, t, i)$

2. 然后，我不确定 $[M_3, M_3](t)$ 这样一项的估计是否正确，或者说 $\mathcal{Q}_{\mathcal{M}_\ni} V(x, y, t, i)$ 计算是否正确？